Applied Mathematical Sciences, Vol. 6, 2012, no. 120, 5967 - 5971

Characterization of Probability Distributions through Contrast of Order Statistics

A. H. Khan, M. Faizan and M. I. Khan[1]

Department of Statistics and Operations Research
Aligarh Muslim University
Aligarh-202 002, India

Abstract

General form of continuous probability distribution is characterized through conditional expectation of contrast of order statistics, conditioned on a non-adjacent order statistics and some of its deductions are discussed.

Mathematics Subject Classification: 62G30, 62E10

Keywords: Characterization; Conditional expectation; Continuous distributions; Order statistics

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a continuous population having the probability density function $(pdf)\, f(x)$ with the distribution function $(df)\, F(x)$ over the support (α, β) and let $X_{1:n} \leq X_{2:n} \leq \ldots X_{n:n}$ be the corresponding order statistics. Then the conditional pdf of $X_{s:n}$ given $X_{r:n} = x$, $1 \leq r < s \leq n$, is [4]

$$f_{s|r}(y \mid x) = \left[\frac{(n-r)!}{(s-r-1)!(n-s)!}\right] \frac{[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}}{[1-F(x)]^{n-r}} f(y)\ ,\ x \leq y \tag{1.1}$$

[1] Corresponding author’s e-mail address :izhar.stats@gmail.com

Conditional expectations of order statistics are extensively used in characterizing the continuous probability distributions. For a detailed survey one may refer to [1, 3, 6, 7, 8, 9, 10 and 11] amongst others. Distributions have been characterized using conditional spacing conditioned on order statistics by Navarro *et al.* [5] and Khan *et al.* [2]. We in this paper have tried to characterize distributions through contrast of conditional expectation of order statistics, extending the earlier known results.

2. Characterization theorem

Theorem: 2.1: Let X be an absolutely continuous random variable with the df $F(x)$ and the pdf $f(x)$ on the support (α, β), where α and β may be finite or infinite. Then for $1 \le m < r < s \le n$,

$$\sum_{i=r}^{s} c_i E[\{h(X_{l:n})\} \mid X_{m:n} = x] = \frac{1}{a}\sum_{i=r}^{s} c_i \sum_{j=m}^{l-1} \frac{1}{(n-j)},\ l = i-1 \quad ,i \tag{2.1}$$

if and only if

$$F(x) = 1 - e^{-ah(x)},\ x \in (\alpha, \beta),\ a > 0 \tag{2.2}$$

where c_i are real numbers $r \le i \le s$, satisfying $\sum_{i=r}^{s} c_i = 0$, $c_i \ne 0$ for some i and $h(x)$ is a monotonic and differentiable function of x such that $F(x)$ is a df.

Proof: First we will prove (2.2) implies (2.1). We have [1]

$$E[\{h(X_{i:n})\} \mid X_{m:n} = x] = h(x) + \frac{1}{a}\sum_{j=m}^{i-1} \frac{1}{(n-j)} \ .$$

Therefore,

$$\sum_{i=r}^{s} c_i E[\{h(X_{i:n})\} \mid X_{m:n} = x] = \sum_{i=r}^{s} c_i \left[h(x) + \frac{1}{a}\sum_{j=m}^{i-1} \frac{1}{(n-j)} \right]$$

$$= \frac{1}{a}\sum_{i=r}^{s} c_i \sum_{j=m}^{i-1} \frac{1}{(n-j)} \quad \text{, as} \quad \sum_{i=r}^{s} c_i = 0$$

hence the ‘if’ part.

To prove the sufficiency part, let $b = \frac{1}{a}\sum_{i=r}^{s} c_i \sum_{j=m}^{i-1} \frac{1}{(n-j)}$

Then

$$\sum_{i=r}^{s} c_i E[\{h(X_{i:n})\} \mid X_{m:n} = x] = b \tag{2.3}$$

or,

$$\sum_{i=r}^{s} c_i \frac{(n-m)!}{(i-m-1)!(n-i)!}\int_x^{\beta} h(y)[F(y)-F(x)]^{i-m-1}[1-F(y)]^{n-i} f(y)dy$$
$$= b[1-F(x)]^{n-m} \tag{2.4}$$

Integrating left hand side of (2.4) by parts treating $[1-F(y)]^{n-i} f(y)$ for integration and $h(y)[F(y)-F(x)]^{i-m-1}$ for differentiation, we get

$$\sum_{i=r}^{s} c_i \frac{(n-m)!}{(i-m-2)!(n-i+1)!}\int_x^{\beta} h(y)[F(y)-F(x)]^{i-m-2}[1-F(y)]^{n-i+1} f(y)dy$$
$$+\sum_{i=r}^{s} c_i \frac{(n-m)!}{(i-m-1)!(n-i+1)!}\int_x^{\beta} h'(y)[F(y)-F(x)]^{i-m-1}[1-F(y)]^{n-i+1} dy$$
$$= b[1-F(x)]^{n-m} \tag{2.5}$$

We can write equation (2.5) as

$$\sum_{i=r}^{s} c_i \frac{(n-m)!}{(i-m-1)!(n-i+1)!}\int_x^{\beta} h'(y)[F(y)-F(x)]^{i-m-1}[1-F(y)]^{n-i+1} dy$$
$$= (b-b_1)[1-F(x)]^{n-m} \quad \text{in view of (2.1)} \tag{2.6}$$

where $b_1 = \frac{1}{a}\sum_{i=r}^{s} c_i \sum_{j=m}^{i-2}\frac{1}{(n-j)}$.

That is,

$$\sum_{i=r}^{s} c_i \frac{(n-m)!}{(i-m-1)!(n-i+1)!}\int_x^{\beta} h'(y)[F(y)-F(x)]^{i-m-1}[1-F(y)]^{n-i+1} dy$$
$$= \frac{1}{a}\sum_{i=r}^{s} c_i \frac{1}{(n-i+1)}[1-F(x)]^{n-m} \tag{2.7}$$

Differentiating (2.7) $(i-m)$ times both sides $w.r.t\ x$, we get

$$\sum_{i=r}^{s} c_i \frac{[1-F(x)]^{n-i}}{(n-i+1)!}[1-F(x)]h'(x) = \sum_{i=r}^{s} c_i \frac{[1-F(x)]^{n-i}}{(n-i+1)!}\frac{f(x)}{a}$$

$$\left[[1-F(x)]h'(x)-\frac{f(x)}{a}\right]\sum_{i=r}^{s} c_i \frac{[1-F(x)]^{n-i}}{(n-i+1)!} = 0$$

$$[1-F(x)]h'(x) = \frac{f(x)}{a} \quad , \text{ as } \sum_{i=r}^{s} c_i \frac{[1-F(x)]^{n-i}}{(n-i+1)!} \neq 0$$

That is,

$$F(x) = 1-\exp[-ah(x)]$$

and hence the Theorem.

Remark 2.1: Putting $c_s = 1$ and $c_r = -1$ in the Theorem 2.1, we get characterizing results as obtained by Khan *et al*. [2] and at $m = r$, we get the result as obtained by Khan and Abouammoh [1].

Table 2.1: Examples based on the distribution function $F(x) = 1 - e^{-a\,h(x)}$, $a > 0$

Distribution	$F(x)$	a	$h(x)$
Exponential	$1 - e^{-\theta x}$ $0 < x < \infty$	θ	x
Weibull	$1 - e^{-\theta x^{p}}$ $0 < x < \infty$	θ	x^{p}
Pareto	$1 - \left(\frac{x}{a}\right)^{-p}$ $a < x < \infty$	p	$\log\left(\frac{x}{a}\right)$
Lomax	$1 - (1 + x)^{-k}$ $0 < x < \infty$	k	$\log(1 + x)$
Gompertz	$1 - \exp[-\frac{\lambda}{\mu}(e^{\mu x} - 1)]$ $0 < x < \infty$	$\frac{\lambda}{\mu}$	$e^{\mu x} - 1$
Beta of the I kind	$1 - (1 - x)^{p}$ $0 < x < 1$	p	$-\log(1 - x)$
Beta of the II kind	$1 - (1 + x)^{-1}$ $0 < x < \infty$	1	$\log(1 + x)$
Extreme value I	$1 - \exp[-e^{x}]$ $-\infty < x < \infty$	1	e^{x}
Log logistic	$1 - (1 + x^{c})^{-1}$ $0 < x < \infty$	1	$\log(1 + x^{c})$
Burr Type IX	$1 - \left[\frac{c\{(1+e^{x})^{k} - 1\}}{2} + 1\right]^{-1}$ $-\infty < x < \infty$	1	$\log\left[\frac{c\{(1+e^{x})^{k} - 1\}}{2} + 1\right]$
Burr Type XII	$1 - (1 + x^{c})^{-k}$ $0 < x < \infty$	k	$\log(1 + x^{c})$

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Received: June, 2012